Canonical Energy-Momentum Tensor of Non-Abelian Fields

Peir-Ru Wang[1,✉]

[1]Department of Physics, National Tsing Hua University, 30013 Hsinchu, Taiwan
✉louisnthu@gapp.nthu.edu.tw

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Abstract

In this article, we provide the natural derivation of symmetrical, gauge-invariant canonical energy-momentum tensor for non-abelian gauge field, i.e., the Yang-Mills theory.

Introduction

As we previously derive the symmetrical, gauge invariant canonical energy-momentum tensor for abelian gauge field [1], in this article we generalize to non-abelian gauge field. A more detailed derivation can be found in the supplementary derivation.

Derivation

We denote p is a point in spacetime, $\mathbf{B}(p)$ as Lie algebra valued gauge potential 1-form of the Yang-Mills field, and the field strength is $\mathbf{G} \equiv d\mathbf{B} + [\mathbf{B} \wedge \mathbf{B}]$. We will discuss the effect of the variation on gauge potential and the spacetime variation. We denote the variation on gauge potential, $\mathbf{B} \to \tilde{\mathbf{B}} = \mathbf{B} + \delta\mathbf{B}$, and the spacetime variation drag by a vector field δx denote as:

$$p \to \tilde{p} = f_{\delta x}(p)$$

The total variation of gauge 1-form is

$$\Delta\mathbf{B} = \tilde{\mathbf{B}}(\tilde{p}) - \mathbf{B}(p) = \delta\mathbf{B} + \hat{\mathcal{L}}_{\delta x}\mathbf{B}$$

In local coordinate $p \to \{x^\mu\}$, the expressions are

$$\mathbf{B}(p) \to B^a_\mu(x^\gamma)\hat{T}_a$$

where $\hat{T}_a$ is the generator of Lie algebra. The field strength in local coordinate

$$\begin{aligned} \mathbf{G} \to G_{\mu\nu} &= \hat{T}_a \partial_\mu B^a_\nu - \hat{T}_a \partial_\nu B^a_\mu + i\lambda[B^a_\mu \hat{T}_a, B^b_\nu \hat{T}_b] \\ &= \hat{T}_a \left(F^a_{\mu\nu} - \lambda f^a_{bc} B^b_\mu B^c_\nu \right) \end{aligned} \tag{1}$$

where

$$[\hat{T}_a, \hat{T}_b] = i f^c_{ab} \hat{T}_c$$

and

$$G^a_{\mu\nu} = F^a_{\mu\nu} - \lambda f^c_{ab} B^a_\mu B^B_\nu$$

. f^c_{ab} is the stgructure constant of Lie algebra. The local coordinate representation of variations are

$$p \to x^\nu = x^\nu + \delta x^\nu$$

$$\Delta B^a_\mu = \delta B^a_\mu + \partial_\nu B^a_\mu \delta x^\nu + B^a_\nu \partial_\mu \delta x^\nu$$

The Lagrangian $\mathscr{L}$ of Yang-Mills field is

$$\mathscr{L} = Tr\left(-\frac{1}{16\pi c} g^{\mu\alpha} g^{\nu\beta} G_{\mu\nu} G_{\alpha\beta}\sqrt{-g}\right) = -\frac{1}{16\pi c} K_{ab} g^{\mu\alpha} g^{\nu\beta} G^a_{\mu\nu} G^b_{\alpha\beta}\sqrt{-g} \tag{2}$$

Here, we denote the Killing form/metric as

$$K_{ab} = Tr(f^c_{ad} f^d_{be}) = f^c_{ad} f^d_{bc}$$

The action S

$$S = \int d^4x\, \mathscr{L}[B^a_\nu(x^\gamma), B^a_{\nu,\mu}(x^\gamma), x^\gamma]$$

We derive the equation of motion (EoM) and the Noether theorem follow the standard procedure [1]. The variation of action ΔS divides into two terms:

$$\Delta S = \int \Delta d^4x * \mathscr{L} + \int d^4x * \Delta\mathscr{L}$$

The first term is the variation of the volume form, which is

$$\Delta d^4x = \partial_\gamma \delta x^\gamma \cdot d^4x$$

The second term is the variation of $\mathscr{L}$

$$\begin{aligned}\Delta\mathscr{L} &= \mathscr{L}[\tilde{B}^a_\nu(\tilde{x}^\gamma), \tilde{B}^a_{\nu,\mu}(\tilde{x}^\gamma), \tilde{x}^\gamma] - \mathscr{L}[B^a_\nu(x^\gamma), B^a_{\nu,\mu}(x^\gamma), x^\gamma] \\ &= \left[\frac{\partial\mathscr{L}}{\partial B^a_\nu}\delta B^a_\nu + \frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\delta(\partial_\mu B^a_\nu)\right](x^\gamma) + \left[\mathscr{L}_{,\gamma}\delta x^\gamma\right](x^\gamma) + O(\delta^2)\end{aligned} \tag{3}$$

Hence the ΔS is

$$\Delta S = \int \left[\frac{\partial\mathscr{L}}{\partial B^a_\nu} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\right)\right]\delta B^a_\nu d^4x + \int \left[\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\delta B^a_\nu\right) + (\mathscr{L}\delta x^\gamma)_{,\gamma}\right] d^4x \tag{4}$$

The EoM is

$$\frac{\partial\mathscr{L}}{\partial B^a_\nu} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\right) = 0$$

Using $\delta B^a_\nu = \Delta B^a_\nu - B^a_{\nu,\gamma}\delta x^\gamma - B^a_\gamma \delta x^\gamma_{,\nu}$:

$$\begin{aligned}\Delta S &= \int \{EoM\}\delta B^a_\nu d^4x + \int \partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}(\Delta B^a_\nu - B^a_{\nu,\gamma}\delta x^\gamma - B^a_\gamma \delta x^\gamma_{,\nu}) + \delta^\mu_\gamma \mathscr{L}\delta x^\gamma\right] d^4x \\ &= \int \{EoM\}\delta B^a_\nu d^4x + \int \underbrace{\partial_\mu}_{(*)}\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\Delta B^a_\nu - \left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}B^a_{\nu,\gamma}\delta x^\gamma + \underbrace{\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}B^a_\gamma \delta x^\gamma_{,\nu}}_{(*)} - \delta^\mu_\gamma \mathscr{L}\delta x^\gamma\right)\right] d^4x\end{aligned}$$

Evaluate the $(*)$ term:

$$\underbrace{\partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}B^a_\gamma \delta x^\gamma_{,\nu}\right]}_{(*)} = \underbrace{\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}B^a_\gamma \delta x^\gamma\right]_{,\nu\mu}}_{(*1)} - \partial_\mu\left[\underbrace{\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\right)_{,\nu} B^a_\gamma \delta x^\gamma}_{(*2)}\right] - \partial_\mu\left[\underbrace{\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}B^a_{\gamma,\nu}\delta x^\gamma}_{(*3)}\right] \tag{5}$$

We first calculate $\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}$ for later use:

$$\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)} = -\frac{1}{4\pi c} K_{ab} g^{\alpha\mu} g^{\beta\nu} G^b_{\alpha\beta}\sqrt{-g}$$

The $(*1)$ term term:

$$\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)} B^a_\gamma \delta x^\gamma\right)_{,\nu\mu} = \left(-\frac{1}{4\pi c} K_{ab} g^{\alpha\mu} g^{\beta\nu} G^b_{\alpha\beta} \sqrt{-g} B^a_\gamma \delta x^\gamma\right)_{,\nu\mu} = 0 \tag{6}$$

due to the antisymmetric of $F_{\mu\nu}$ and the symmetric of second order derivative $\{_{,\nu\mu}\}$.

The $(*2)$ term is:

$$\underbrace{\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\right)_{,\nu} B^a_\gamma \delta x^\gamma}_{(*2)} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)} \left(\lambda f^a_{bn} B^n_\nu B^b_\gamma \delta x^\gamma\right) \tag{7}$$

We now have:
Hence $(*)$ becomes:

$$\underbrace{\partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)} B^a_\gamma \delta x^\gamma_{,\nu}\right]}_{(*)} = -\partial_\mu \left[\underbrace{\frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)} \lambda f^a_{bn} B^b_\gamma B^n_\nu \delta x^\gamma}_{(*2)}\right] - \partial_\mu \left[\underbrace{\frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)} B^a_{\gamma,\nu} \delta x^\gamma}_{(*3)}\right]$$

$$\begin{aligned} \Delta S &= \int \{EoM\} \delta B^a_\nu \, d^4x + \int \partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)} \Delta B^a_\nu - \frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)} \left(B^a_{\nu,\gamma} - \underbrace{B^a_{\gamma,\nu}}_{(*3)} - \underbrace{\lambda f^a_{bn} B^b_\gamma B^n_\nu}_{(*2)}\right) \delta x^\gamma + \delta^\mu_\gamma \mathscr{L} \delta x^\gamma\right] d^4x \\ &= \int \{EoM\} \delta B^a_\nu \, d^4x + \int \partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)} \Delta B^a_\nu - \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)} G^a_{\gamma\nu} - \delta^\mu_\gamma \mathscr{L}\right) \delta x^\gamma\right] d^4x \end{aligned}$$

Hence we have

$$\begin{aligned} T^\mu_\gamma &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)} G^a_{\gamma\nu} - \delta^\mu_\gamma \mathscr{L} \\ &= \left(-\frac{1}{4\pi c} K_{ab} g^{\alpha\mu} g^{\beta\nu} G^b_{\alpha\beta} \sqrt{-g}\right) G^a_{\gamma\nu} - \delta^\mu_\gamma \left(-\frac{1}{16\pi c} K_{ab} g^{\mu\alpha} g^{\nu\beta} G^a_{\mu\nu} G^b_{\alpha\beta} \sqrt{-g}\right) \\ &= -\frac{1}{4\pi c} G^{\mu\nu}_a G^a_{\gamma\nu} \sqrt{-g} + \delta^\mu_\gamma \frac{1}{16\pi c} G^{\alpha\beta}_a G^a_{\alpha\beta} \sqrt{-g} \end{aligned}$$

Summary

The natural symmetrical, gauge-invariant canonical energy-momentum tensor for the non-abelian gauge field is derived. This derivation does not depend on flat spacetime geometry, hence is background independent. This method has potential to cover general relativity.

References

1. Canonical Energy-Momentum Tensor of Abelian Fields, arXiv:2503.15031.

Supplementary Derivation

Eq.(1)
The derivation of Eq.(1)

$$\begin{aligned}\mathbf{G}\to G_{\mu\nu} &= \hat{T}_a\partial_\mu B^a_\nu - \hat{T}_a\partial_\nu B^a_\mu + i\lambda[B^a_\mu\hat{T}_a, B^b_\nu\hat{T}_b]\\ &= \hat{T}_a\partial_\mu B^a_\nu - \hat{T}_a\partial_\nu B^a_\mu + i\lambda B^a_\mu B^b_\nu[\hat{T}_a,\hat{T}_b]\\ &= \hat{T}_a\left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu - \lambda f^a_{bc}B^b_\mu B^c_\nu\right)\\ &= \hat{T}_a\left(F^a_{\mu\nu} - \lambda f^a_{bc}B^b_\mu B^c_\nu\right)\end{aligned}$$

Eq.(2)
The explicit expression of Lagrangian Eq.(2) is

$$K_{ab}g^{\mu\alpha}g^{\nu\beta}G^a_{\mu\nu}G^b_{\alpha\beta} = K_{ab}g^{\mu\alpha}g^{\nu\beta}\left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu - \lambda f^a_{cd}B^c_\mu B^d_\nu\right)\left(\partial_\alpha B^b_\beta - \partial_\beta B^b_\alpha - \lambda f^b_{ef}B^e_\alpha B^f_\beta\right)$$

Eq.(3)
The derivation of $\Delta\mathscr{L}$ is

$$\begin{aligned}\Delta\mathscr{L} &= \mathscr{L}[\tilde{B}^a_\nu(\tilde{x}^\gamma), \tilde{B}^a_{\nu,\mu}(\tilde{x}^\gamma), \tilde{x}^\gamma] - \mathscr{L}[B^a_\nu(x^\gamma), B^a_{\nu,\mu}(x^\gamma), x^\gamma]\\ &= \mathscr{L}[\tilde{B}^a_\nu(\tilde{x}^\gamma), \tilde{B}^a_{\nu,\mu}(\tilde{x}^\gamma), \tilde{x}^\gamma] - \mathscr{L}[B^a_\nu(\tilde{x}^\gamma), B^a_{\nu,\mu}(\tilde{x}^\gamma), \tilde{x}^\gamma] + \mathscr{L}[B^a_\nu(\tilde{x}^\gamma), B^a_{\nu,\mu}(\tilde{x}^\gamma), \tilde{x}^\gamma] - \mathscr{L}[B^a_\nu(x^\gamma), B^a_{\nu,\mu}(x^\gamma), x^\gamma]\\ &= \left[\frac{\partial\mathscr{L}}{\partial B^a_\nu}\delta B^a_\nu + \frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\delta(\partial_\mu B^a_\nu)\right](\tilde{x}^\gamma) + \left[\mathscr{L}_{,\gamma}\delta x^\gamma\right](x^\gamma)\\ &= \left[\frac{\partial\mathscr{L}}{\partial B^a_\nu}\delta B^a_\nu + \frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\delta(\partial_\mu B^a_\nu)\right](x^\gamma) + O(\delta^2) + \left[\mathscr{L}_{,\gamma}\delta x^\gamma\right](x^\gamma)\end{aligned}$$

Eq.(4)
The derivation of ΔS is

$$\begin{aligned}\Delta S &= \int d^4x\cdot\mathscr{L}\delta x^\gamma_{,\gamma} + \int d^4x\left[\frac{\partial\mathscr{L}}{\partial B^a_\nu}\delta B^a_\nu + \frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\delta(\partial_\mu B^a_\nu) + \mathscr{L}_{,\gamma}\delta x^\gamma\right]\\ &= \int d^4x\cdot\left[\frac{\partial\mathscr{L}}{\partial B^a_\nu}\delta B^a_\nu + \frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\partial_\mu(\delta B^a_\nu) + (\mathscr{L}\delta x^\gamma)_{,\gamma}\right]\\ &= \int\left[\frac{\partial\mathscr{L}}{\partial B^a_\nu} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\right)\right]\delta B^a_\nu d^4x + \int\left[\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\delta B^a_\nu\right) + (\mathscr{L}\delta x^\gamma)_{,\gamma}\right]d^4x\end{aligned}$$

Eq.(5)

$$\begin{aligned}\underbrace{\partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}B^a_\gamma\delta x^\gamma_{,\nu}\right]}_{(*)} &= \partial_\mu\left[\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}B^a_\gamma\delta x^\gamma\right)_{,\nu} - \left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\right)_{,\nu}B^a_\gamma\delta x^\gamma - \frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}B^a_{\gamma,\nu}\delta x^\gamma\right]\\ &= \underbrace{\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}B^a_\gamma\delta x^\gamma\right]_{,\nu\mu}}_{(1)} - \underbrace{\partial_\mu\left[\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\right)_{,\nu}B^a_\gamma\delta x^\gamma\right]}_{(2)} - \underbrace{\partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}B^a_{\gamma,\nu}\delta x^\gamma\right]}_{(3)}\end{aligned}$$

Eq.(6)

$$\begin{aligned}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu B^a_\nu)}B^a_\gamma\delta x^\nu\right)_{,\nu\mu} &= \left(-\frac{1}{4\pi c}K_{ab}g^{\alpha\mu}g^{\beta\nu}G^b_{\alpha\beta}\sqrt{-g}B^a_\gamma\delta x^\nu\right)_{,\nu\mu}\\ &= \left(-\frac{1}{8\pi c}K_{ab}g^{\alpha\mu}g^{\beta\nu}G^b_{\alpha\beta}\sqrt{-g}B^a_\gamma\delta x^\nu\right)_{,\nu\mu} + \left(-\frac{1}{8\pi c}K_{ab}g^{\alpha\mu}g^{\beta\nu}G^b_{\alpha\beta}\sqrt{-g}B^a_\gamma\delta x^\nu\right)_{,\nu\mu}\\ &= \left(-\frac{1}{8\pi c}K_{ab}g^{\alpha\mu}g^{\beta\nu}G^b_{\alpha\beta}\sqrt{-g}B^a_\gamma\delta x^\nu\right)_{,\nu\mu} + \left(-\frac{1}{8\pi c}K_{ab}g^{\beta\nu}g^{\alpha\mu}G^b_{\beta\alpha}\sqrt{-g}B^a_\gamma\delta x^\nu\right)_{,\mu\nu}\\ &= \left(-\frac{1}{8\pi c}K_{ab}g^{\alpha\mu}g^{\beta\nu}G^b_{\alpha\beta}\sqrt{-g}B^a_\gamma\delta x^\nu\right)_{,\nu\mu} - \left(-\frac{1}{8\pi c}K_{ab}g^{\alpha\mu}g^{\beta\nu}G^b_{\alpha\beta}\sqrt{-g}B^a_\gamma\delta x^\nu\right)_{,\nu\mu} = 0\end{aligned}$$

Eq.(7)
The $(*2)$ term rely on the EoM:

$$\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\right)_{,\mu} = \frac{\partial \mathscr{L}}{\partial B^a_\nu}$$

$\left(\frac{\partial \mathcal{L}}{\partial(\partial_\phi B^a_\varepsilon)}\right)_{,\phi} = \left(-\frac{1}{4\pi c}K_{ab}g^{\alpha\phi}g^{\beta\varepsilon}G^b_{\alpha\beta}\sqrt{-g}\right)_{,\phi} = \frac{\partial \mathcal{L}}{\partial B^a_\varepsilon}$ is EoM
$\left(\frac{\partial \mathcal{L}}{\partial(\partial_\varepsilon B^a_\phi)}\right)_{,\phi} = \left(-\frac{1}{4\pi c}K_{ab}g^{\alpha\varepsilon}g^{\beta\phi}G^b_{\alpha\beta}\sqrt{-g}\right)_{,\phi} = \left(-\frac{1}{4\pi c}K_{ab}g^{\beta\varepsilon}g^{\alpha\phi}G^b_{\beta\alpha}\sqrt{-g}\right)_{,\phi} = -\left(-\frac{1}{4\pi c}K_{ab}g^{\alpha\varepsilon}g^{\beta\phi}G^b_{\alpha\beta}\sqrt{-g}\right)_{,\phi} = -\frac{\partial \mathcal{L}}{\partial B^a_\varepsilon}$

The explicit form of $\frac{\partial \mathscr{L}}{\partial B^a_\mu}$ is:

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial B^k_\varepsilon} &= -\frac{1}{8\pi c}K_{ab}g^{\mu\alpha}g^{\nu\beta}G^a_{\alpha\beta}\sqrt{-g}\frac{\partial}{\partial B^k_\varepsilon}\left(F^b_{\mu\nu} - \lambda f^b_{mn}B^m_\mu B^n_\nu\right) \\
&= -\left(-\frac{1}{4\pi c}K_{ab}g^{\mu\alpha}g^{\nu\beta}G^a_{\alpha\beta}\sqrt{-g}\right)\left(\lambda f^b_{mn}\delta^\varepsilon_\mu\delta^m_k B^n_\nu - \lambda f^b_{mn}B^m_\mu\delta^\varepsilon_\nu\delta^n_k\right) \\
&= -\left(-\frac{1}{4\pi c}K_{ab}g^{\mu\alpha}g^{\nu\beta}G^a_{\alpha\beta}\sqrt{-g}\right)\left(\lambda f^b_{kn}\delta^\varepsilon_\mu B^n_\nu - \lambda f^b_{mk}B^m_\mu\delta^\varepsilon_\nu\right) \\
&= -\left(-\frac{1}{4\pi c}K_{ab}g^{\mu\alpha}g^{\nu\beta}G^a_{\alpha\beta}\sqrt{-g}\right)\left(\lambda f^b_{kn}\delta^\varepsilon_\mu B^n_\nu + \lambda f^b_{kn}B^n_\mu\delta^\varepsilon_\nu\right) \\
&= -\left(-\frac{1}{4\pi c}K_{ab}g^{\mu\alpha}g^{\nu\beta}G^a_{\alpha\beta}\sqrt{-g}\right)\lambda f^b_{kn}\left(\delta^\varepsilon_\mu B^n_\nu + \delta^\varepsilon_\nu B^n_\mu\right) \\
&= -\frac{\partial \mathscr{L}}{\partial(\partial_\varepsilon B^b_\nu)}\lambda f^b_{kn}\left(\delta^\varepsilon_\mu B^n_\nu + \delta^\varepsilon_\nu B^n_\mu\right)
\end{aligned}$$

$$\frac{\partial \mathscr{L}}{\partial B^k_\varepsilon}B^k_\gamma = -\frac{\partial \mathscr{L}}{\partial(\partial_\varepsilon B^b_\nu)}\lambda f^b_{kn}\left(\delta^\varepsilon_\mu B^n_\nu + \delta^\varepsilon_\nu B^n_\mu\right)B^k_\gamma$$

The $(*2)$ term is:

$$\begin{aligned}
\underbrace{\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\right)_{,\nu} B^a_\gamma \delta x^\gamma}_{(*2)} &= -\frac{\partial \mathcal{L}}{\partial B^a_\mu}B^a_\gamma\delta x^\gamma \\
&= \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu B^b_\nu)}(\lambda f^b_{an}B^n_\nu)\right]B^a_\gamma\delta x^\gamma \\
&= \frac{\partial \mathscr{L}}{\partial(\partial_\mu B^a_\nu)}\left(\lambda f^a_{bn}B^n_\nu B^b_\gamma\delta x^\gamma\right)
\end{aligned}$$